The Discrete Fourier Transform of Symmetric Sequences

Symmetric sequences arise often in digital signal processing. Examples include symmetric pulses, window functions, and the coefficients of most finite-impulse response (FIR) filters, not to mention the cosine function. Examining symmetric sequences can give us some insights into the Discrete Fourier Transform (DFT). An even-symmetric sequence is centered at n = 0 and $x_{even}(n) = x_{even}(-n)$. The DFT of $x_{even}(n)$ is real. Most often, signals we encounter start at n = 0, so they are not strictly speaking even-symmetric. We'll look at the relationship between the DFT's of such sequences and those of true even-symmetric sequences. Note: for basics of using the DFT, see my last post [1].

Let x(n) be a causal sequence as shown in Figure 1 (top). Let $x_{even}(n)$ be an even-symmetric version of x(n), defined over n = -8:7, as shown in Figure 1 (bottom). This sequence is centered at n = 0, and the first non-zero value occurs at n = -3. The sequence is also referred to as a *non-causal* sequence, because it begins before n = 0. Mathematically, the most straightforward way to find the Discrete Fourier Transform (DFT) of this sequence would be to evaluate the DFT formula (see Appendix) over n = -8: 7. We would then find that the spectrum $X_{even}(k)$ is real. However, in this article, we'll compute the DFT using the standard time index range of n= 0: N-1, which allows us to use the Matlab Fast Fourier Transform (FFT) function. We'll find $X_{even}(k)$ using two different methods.

Method 1: Time Shift

Given the causal sequence x(n), we can use the *time-shifting property* of the DFT to find the DFT of $x_{even}(n)$. For x(n) with DFT X(k), the time-shifting property is given by (see Appendix) :

$$x(n-N_0) \underset{DFT}{\longleftrightarrow} e^{-j2\pi N_0 k/N} X(k) \qquad (1a)$$

Where X(k) is the DFT of x(n) and N_0 is delay in samples. We define normalized radian frequency $\omega = 2\pi f/f_s$, where f_s is sample frequency in Hz and $f = kf_s/N$. We can then also write:

$$x(n-N_0) \underset{DFT}{\longleftrightarrow} e^{-j\omega N_0} X(\omega) \qquad (1b)$$

Consider x(n) and $x_{even}(n)$ shown in Figure 1. $x_{even}(n)$ is equal to x(n) advanced in time by N_0 = 3 samples, so:

$$x_{even}(n) = x(n+N_0) \qquad (2)$$

Since we are *advancing* x(n) by N_0 samples, Equation 1b becomes:

$$x_{even}(n) = x(n+N_0) \underset{DFT}{\longleftrightarrow} e^{j\omega N_0} X(\omega) \qquad (3)$$

Thus, the DFT of $x_{even}(n)$ is:

$$X_{even}(\omega) = e^{j\omega N_0} X(\omega) \qquad (4)$$

We can also write the converse of Equation 4:

$$X(\omega) = e^{-j\omega N_0}\, X_{even}(\omega) \qquad (5)$$

This equation shows that the DFT of a sequence x(n) having even symmetry with respect to its center sample is a real spectrum $X_{even}(\omega)$ multiplied by a linear phase shift. An example of this is the frequency response of a symmetric FIR filter with an odd number of taps. Given an even-symmetric filter $h_{even}(n)$ with real frequency response $H_{even}(\omega)$, the causal filter's frequency response is linear-phase:

$$H(\omega) = e^{-j\omega N_0}\, H_{even}(\omega) \qquad (6)$$

where N_0 = (number of taps – 1)/2. A symmetric FIR with an even number of taps also has linear phase [2].

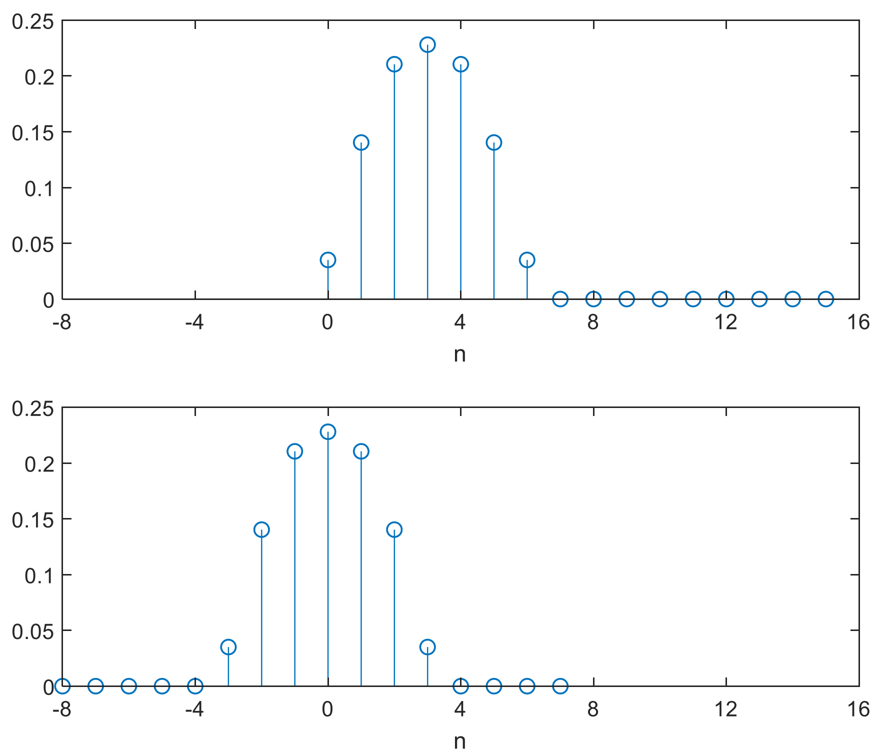

Figure 1. Top: Causal sequence x(n). Bottom: Even-symmetric sequence $x_{even}(n)$.

Method 1 Example

In this example, we use Equation 4 to find the DFT of $x_{even}(n)$ shown in Figure 1 (bottom), given the causal sequence x(n) of Figure 1 (top):

```
x(n) = [2 8 12 13 12 8 2 0 0 0 0 0 0 0 0 0]/57.
```

The Matlab code is listed below. Note that the .* operator performs element-by-element multiplication of two vectors.

```
fs= 1;              % Hz sample frequency
N= 16;              % samples length of x
x= [2 8 12 13 12 8 2 0 0 0 0 0 0 0 0 0]/57;  % causal sequence
% compute DFT of causal x
X= fft(x,N);         % DFT
k= 0:N-1;            % frequency index
f= k*fs/N;           % Hz  frequency

% compute DFT of x_even using time shift property of DFT
w= 2*pi*f/fs;               % rad normalized radian frequency
No = 3;                     % samples  time advance
Xeven= exp(j*w*No).*X;      % Equation 4
```

The DFT of x(n) is plotted in Figure 2; we see that it is complex. The DFT of $x_{even}(n)$ is plotted in Figure 3; as expected, it is real.

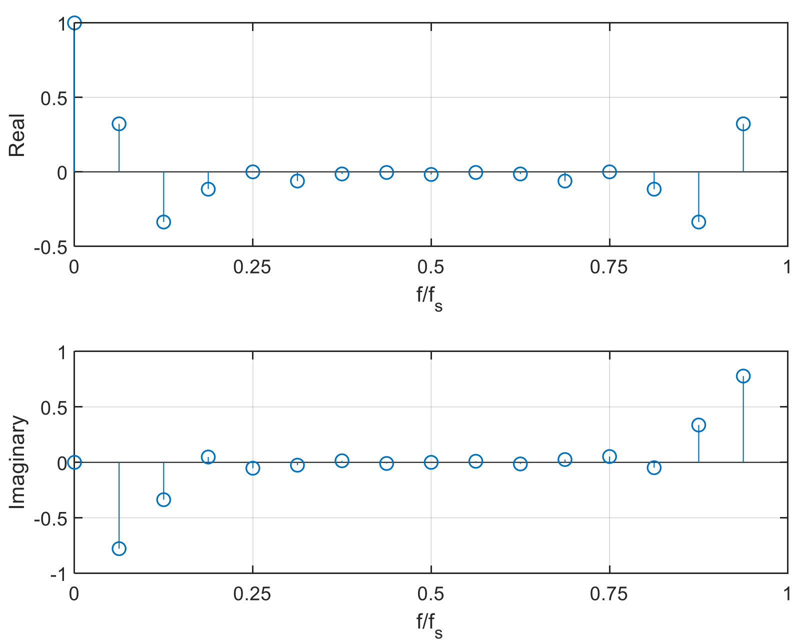

Figure 2. DFT of causal sequence x(n). Top: real part. Bottom: imaginary part.

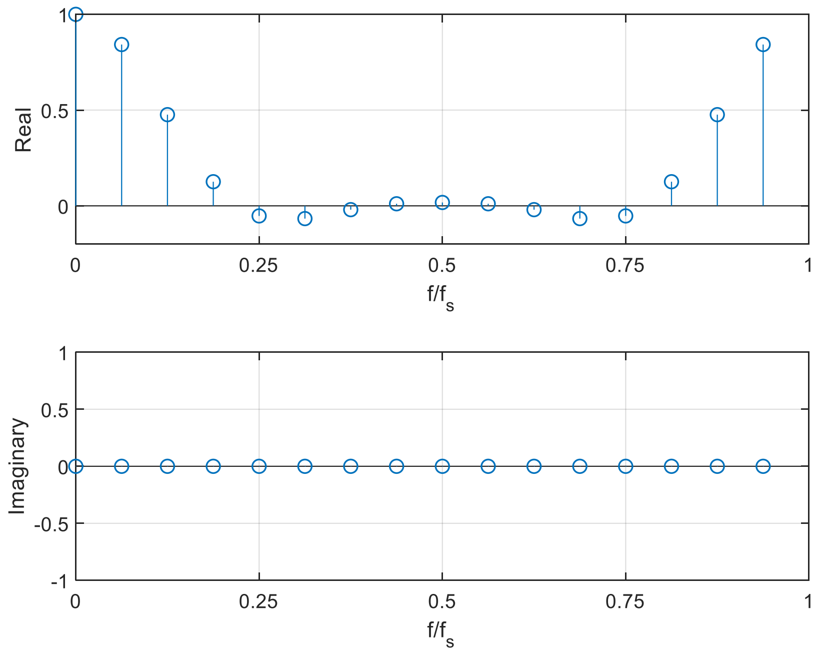

Figure 3. DFT of $x_{even}(n)$. Top: real part. Bottom: imaginary part.

Method 2: Periodic Extension in n

Figure 1 (bottom) plots $x_{even}(n)$, which has finite length N = 16 samples. Its spectrum, which we computed using the DFT, is of course discrete, as shown in Figure 3. You may recall that the Fourier Transform of a periodic signal is discrete. The converse is also true: the inverse Fourier Transform of a discrete spectrum is periodic. So, mathematically, our finite-length $x_{even}(n)$ can be viewed as periodic, with each period replicating its N samples [3]. This is shown in Figure 4, where the top plot shows $x_{even}(n)$, and the center plot shows $x_{even}(n)$ extended to be periodic.

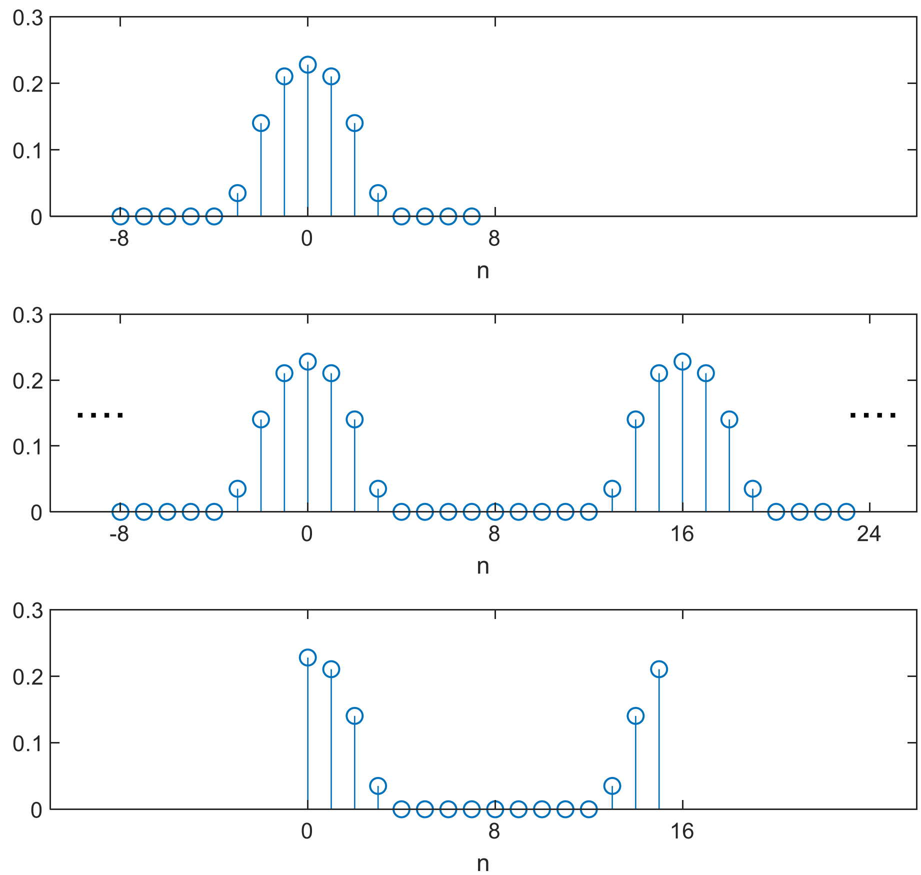

Figure 4. Top: sequence $x_{even}(n)$. Middle: periodic extension $x_p(n)$. Bottom: $u(n) = x_p(0{:}N\text{-}1)$.

For our periodic sequence $x_p(n)$ we can state:

$$x_p(n+N) = x_p(n) \qquad (7)$$

Thus,

$$x_p(N-1) = x_p(-1)$$
$$x_p(N-2) = x_p(-2) \quad etc. \qquad (8)$$

If we define $u(n)= x_p(0{:}N\text{-}1)$, then u(n) is as shown in Figure 4 (bottom). Conveniently, the time index n of u(n) matches that used in the DFT formula (see Appendix). Note that u(n) has even symmetry with respect to N/2 = 8 (not including the sample at N = 0). The DFT of u(n) is real, as we'll show in the following example.

Method 2 Example

Here is the Matlab code to find u(n) given $x_{even}(n)$, and compute its DFT.

```
fs= 1;              % Hz sample frequency
N= 16;              % samples  length of x_even
x_even= [0 0 0 0 0 2 8 12 13 12 8 2 0 0 0 0]/57;

xp= [x_even x_even];    % periodic extension of x_even (2 periods)
u= xp(9:24);            % u = xp over n= 0:N-1

U= fft(u,N);        % DFT
k= 0:N-1;           % frequency index
f= k*fs/N;          % Hz  frequency
```

x_even, xp, and u are plotted in Figure 4. The DFT of u(n) is real and identical to the DFT we computed in Example 1; see Figure 3.

From Equation 8, $x_p(N/2: N-1) = x_p(-N/2: -1)$. That is, the samples of x_p from N/2: N-1 match the negative-time portion of x_p. So, we can view the range n = N/2: N-1 as negative time, and any sequence with non-zero samples in this range is non-causal. Common examples of non-causal sequences are any periodic sequence, such as a cosine.

If we form the bottom plot of u(n) in Figure 4 into a circle, we get the three-dimensional plot of Figure 5. The symmetry with respect to n= 0 or n = N/2 is apparent. The plot shows the equivalence of $x_{even}(n)$ and u(n). The plot can be viewed as periodic, with each period represented by one trip around the circle.

Finally, a word about odd-symmetric sequences. An odd-symmetric sequence is centered at n = 0 and $x_{odd}(n) = -x_{odd}(-n)$. The DFT of such a sequence is pure imaginary. Examples of odd sequences are the coefficients of FIR differentiators [4] and Hilbert transformers.

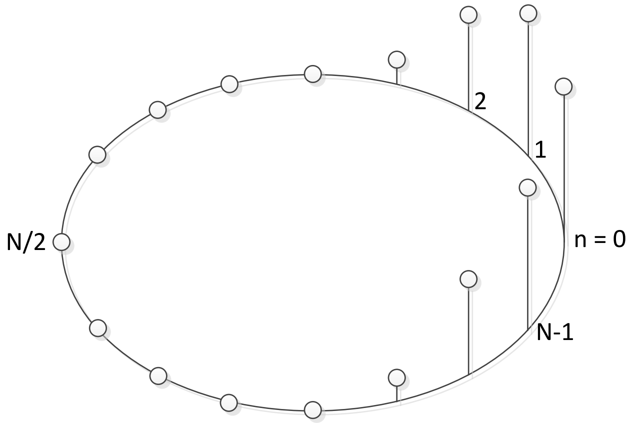

Figure 5. Circular plot of u(n), N = 16.

Appendix: DFT Formula and the DFT Time-shift Property

For a discrete-time sequence x(n), the DFT is defined as:

$$X(k) = \sum_{n=0}^{N-1} \mathrm{x(n)e}^{-\mathrm{j2\pi kn}/N} \qquad (A-1)$$

where

X(k) = discrete frequency spectrum of time sequence x(n)
N = number of samples of x(n) and X(k)
n = 0: N-1 = time index
k = 0: N-1 = frequency index

Equation 1 calculates a single spectral component or *frequency sample* X(k). To find the whole spectrum over k = 0 to N-1, Equation 1 must be evaluated N times.

We see that, by definition, the DFT applies to a finite-length sequence of N samples. Equation 1 does not contain variables for time and frequency, but uses time and frequency indices n and k instead. The frequency index is sometimes referred to as "frequency bins." For sample time of T_s, the discrete time variable is given by:

$$t = nT_s \qquad (A\text{-}2)$$

For sample frequency $f_s = 1/T_s$, the discrete frequency variable is given by:

$$f = k^*f_s/N \qquad (A\text{-}3)$$

While x(n) is normally a real sequence, X(k) is in general complex. For real x(n), the real part of X(k) is even with respect to $f = f_s/2$, and the imaginary part is odd.

Time-Shift Property

Figure A-1 (top) shows a sequence x(n). If we delay x(n) by N_0 samples, we get the sequence:

$$y(n) = x(n - N_0) \qquad (A-4)$$

This sequence is shown in the bottom plot for $N_0 = 2$. Using Equation A-1, we can write the DFT of y(n):

$$Y(k) = \sum_{n=N_0}^{N_0+N-1} \mathrm{x}(\mathrm{n} - N_0)\mathrm{e}^{-\mathrm{j}2\pi\mathrm{kn}/N} \qquad (A-5)$$

Now substitute m = n – N_0 into this equation:

$$Y(k) = \sum_{m=0}^{N-1} \mathrm{x}(\mathrm{m})\mathrm{e}^{-\mathrm{j}2\pi\mathrm{k}(\mathrm{m}+N_0)/N} \qquad (A-6)$$

or,

$$Y(k) = e^{-j2\pi N_0 k/N} \sum_{m=0}^{N-1} \mathrm{x}(\mathrm{m})\mathrm{e}^{-\mathrm{j}2\pi\mathrm{km}/N} \qquad (A-6)$$

Comparing this to Equation A-1, we see that the summation is just X(k), so we have:

$$Y(k) = e^{-j2\pi N_0 k/N} X(k) \qquad (A-7)$$

Thus,

$$x(n - N_0) \underset{DFT}{\longleftrightarrow} e^{-j2\pi N_0 k/N} X(k) \qquad (A-8)$$

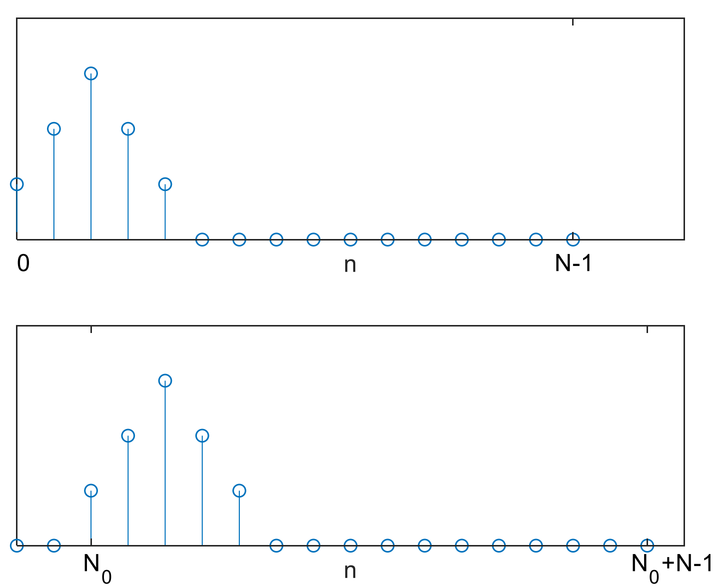

Figure A-1. Top: Sequence x(n). Bottom: Shifted sequence y(n) = x(n – N_0) for N_0 = 2.

References

1. Robertson, Neil, "Learn to Use the Discrete Fourier Transform", DSPRelated.com, Sept, 2024, https://www.dsprelated.com/showarticle/1696.php

2. Mitra, Sanjit K., Digital Signal Processing, 2nd Ed., McGraw Hill, 2001, Section 4.4.3.

3. Lyons, Richard G., Understanding Digital Signal Processing, 3rd Ed., Pearson, 2011, Section 3.14.

4. Robertson, Neil, "Evaluate Noise Performance of Discrete-Time Differentiators", DSPRelated.com, March, 2022, https://www.dsprelated.com/showarticle/1447.php

December, 2024 Neil Robertson